

A fun intro to 1D kinematics

P. Fraundorf

*Physics & Astronomy/Center for NanoScience, U. Missouri-StL (63121) and
Physics, Washington University (63110), St. Louis, MO, USA**

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Introductory physics texts often start with: (i) time as an implicitly universal-variable, and (ii) the oddly mass-independent acceleration-due-to-gravity near earth’s surface, as givens with no larger context. In this short note, we take an “engineering” rather than a “physics” approach and (for those who might enjoy it) invoke modern concepts to tell a story about: (a) time as a quantity like position that depends on one’s choice of yardsticks & clocks, and (b) the geometric origins of gravitational acceleration. The note is designed to tantalize students interested in the subject with predictive equations within range of their math-background, while for students otherwise interested it provides context while delivering the good-news that *their* course involves only low-speed approximations.

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Contents

I. introduction	1
II. velocity parameters	1
III. acceleration integrals	2
IV. acceleration-related curvature	2
V. gravity’s acceleration	2
VI. discussion	3
Acknowledgments	3
References	3

I. INTRODUCTION

Intro-physics students in an engineering-physics class might augment their introduction to unidirectional-kinematics with a technical story that is in harmony with the general trend toward metric-first approaches^{1,2}. This might nip the implicit Newtonian-assumption of universal time in the bud. It would also introduce: (i) momentum-proportional proper-velocities^{3,4} that can be added vectorially at any speed⁵, and (ii) proper-acceleration⁶ in harmony with the modern (equivalence-principle based) understanding of geometric-accelerations i.e. accelerations that arise from choice of a non free-float-frame coordinate-system.

In the process of putting into context the scalar Newtonian-concepts of: **reference-frame**, **elapsed-time** t , **position** x , **velocity** v , **acceleration** a , **constant-acceleration-integral** and **gravitational-acceleration** g , this intro is designed to superficially (i.e. without quantitative testing) expose students to more robust technical-concepts highlighted in **bold** below. If these engender critical-discussion as distinct from e.g. naive-obsession with intuition-conflicts, all the better

as such concepts might positively inspire the empirical-scientist inside students even if they never take another physics course.

The few take-home equations from this introduction that you’ll be asked to master in the course will be highlighted in **red**. Hence you might simply consider these notes an alternate, but fun, introduction to a few equations that are typically just handed to students (explicitly or implicitly) with no introduction at all.

II. VELOCITY PARAMETERS

We begin with Minkowski’s (1+1)D space-time version of Pythagoras’ theorem, i.e. the **flat-space metric-equation**, described as an equation for the proper-time τ elapsed on a traveler’s clock:

$$(c\delta\tau_{\text{traveler}})^2 = (c\delta t_{\text{map}})^2 - (\delta x_{\text{map}})^2 \quad (1)$$

in terms of traveler position x and time t as measured on synchronized clocks fixed to yardsticks on an extended reference or map-frame. Here c is the space-time constant known as **lightspeed** i.e. $c = 299,792,458$ meters/second. Lower case δ refers to a small change in a variable (final value minus initial value), while upper case Δ will also work for large changes in that variable.

The metric-equation lets one define relations between three velocity-parameters, namely **coordinate-velocity** $v \equiv dx/dt$, **proper-velocity** $w \equiv dx/d\tau = \gamma v$, and **Lorentz-factor**:

$$\gamma \equiv \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \sqrt{1 + \left(\frac{w}{c}\right)^2} \quad (2)$$

At low speeds the Lorentz-factor γ reaches its lower-limit of $dt/d\tau = 1$, in which case traveler clocks and map clocks stay in sync, while at high speeds coordinate-velocity v has an upper-limit equal to c . Proper-velocity w (also known as momentum per unit mass) has no lower or upper limits.

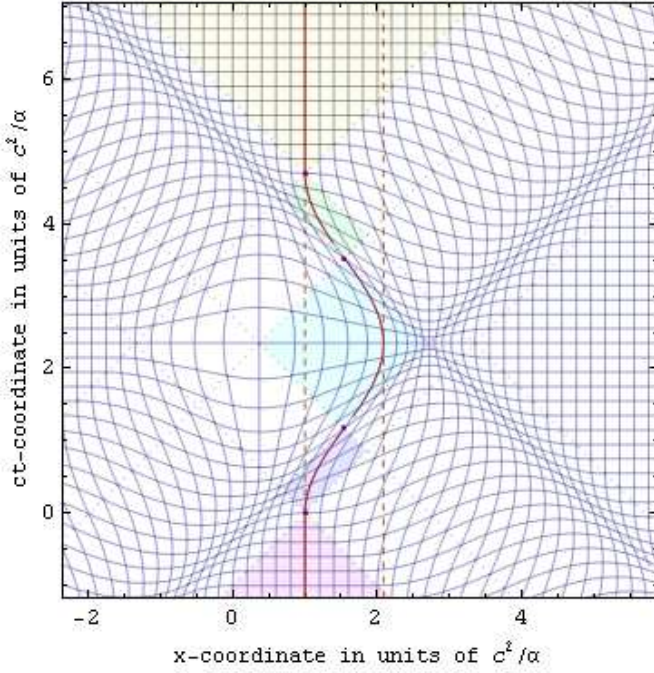


FIG. 1: In special-relativity with radar-time simultaneity⁷, acceleration curves flat spacetime.

Each of these three velocity-parameters is also simply related to the traveling object's hyperbolic-velocity-angle or **rapidity** η by

$$\eta = \sinh^{-1} \left[\frac{w}{c} \right] = \tanh^{-1} \left[\frac{v}{c} \right] = \pm \cosh^{-1} [\gamma]. \quad (3)$$

The good news for students in an engineering physics class is that their applications generally involve velocities v much less than lightspeed c so that coordinate-velocity $v \equiv dx/dt$ is the only velocity-parameter needed i.e. $v \simeq w \simeq \eta c$, plus the Lorentz-factor $\gamma \simeq 1$ so that $\Delta t \simeq \Delta \tau$.

III. ACCELERATION INTEGRALS

One might work toward constant-acceleration integrals at any speed by first defining **coordinate-acceleration** as $a \equiv dv/dt = d^2x/dt^2$.

Unidirectional **proper-acceleration** α (e.g. felt by the traveler as a result of point-of-action contact-forces) can then be written in integral velocity-time/position form as:

$$\alpha = \gamma^3 a = \frac{\Delta w}{\Delta t} = c \frac{\Delta \eta}{\Delta \tau} = c^2 \frac{\Delta \gamma}{\Delta x} \quad (4)$$

which at low speeds i.e. when $v \ll c$ yield the more familiar velocity-time and velocity-position integrals of constant coordinate-acceleration:

$$\alpha \stackrel{v \ll c}{\simeq} a = \frac{\Delta v}{\Delta t} = \frac{1}{2} \frac{\Delta(v^2)}{\Delta x} \quad (5)$$

Position-time versions of these constant-acceleration integrals may then be written in less-compact form e.g. as:

$$\Delta x \stackrel{v \ll c}{\simeq} \frac{c^2}{\alpha} (\cosh [\frac{\alpha \Delta \tau}{c} + \eta_o] - \cosh [\eta_o]) = v_o \Delta t + \frac{1}{2} a (\Delta t)^2. \quad (6)$$

Hence in engineering physics the integrals we need to master are basically $v \simeq v_o + a \Delta t$, $v^2 \simeq v_o^2 + 2a \Delta x$, and $x \simeq v_o t + \frac{1}{2} a (\Delta t)^2$.

IV. ACCELERATION-RELATED CURVATURE

The map-frame ct versus x plot in Fig. 1 then shows how acceleration, in this case of a 1-gee proper-acceleration round-trip lasting 4 traveler-years, distorts distances (blue vertical mesh-lines) and simultaneity (blue horizontal mesh-lines) experienced by that accelerated observer. For objects that are extended along the line of their acceleration, these distortions in space and time will occur even across an accelerated-object's own length.

For example, in addition to the metric-equation's motion-related **time-dilation** in which:

$$\delta \tau_{\text{traveler}} = \delta t_{\text{map}} \sqrt{1 - \left(\frac{v}{c} \right)^2}, \quad (7)$$

for accelerated objects of length L in the direction of proper-acceleration α , one finds an acceleration-related time-dilation of the form:

$$\delta \tau_{\text{trailing}} \simeq \delta \tau_{\text{leading}} \sqrt{1 - \frac{2\alpha L}{c^2}}. \quad (8)$$

Here the leading-edge of the object is in the direction of the acceleration α , not necessarily in the direction of travel.

For the 1-gee proper-acceleration of a standing human in the vertical direction, this **differential-aging** between head and foot becomes $dt_{\text{foot}}/dt_{\text{head}} \simeq 1 - 2 \times 10^{-16}$. This means that if you stand up (or sit tall) for a sizeable fraction of your lifetime, **your head may be a few-hundred nanoseconds older than your feet**. This is a small effect for humans, but as discussed below (and illustrated in Fig. 2) it's quite significant for global-positioning satellites for which nano-second timing-errors give rise to macroscopic errors in position.

V. GRAVITY'S ACCELERATION

Einstein's general-relativity shows how a gravitational-acceleration that is the same for all masses can be seen to result from a mass-related static-distortion of space-time. This can be described most simply with a modified metric-equation of the form:

$$(c \delta \tau_{\text{radius-r}})^2 = (1 - \frac{r_o}{r}) (c \delta t_{\text{far}})^2 - \frac{(\delta x_{\text{far}})^2}{(1 - \frac{r_o}{r})}. \quad (9)$$

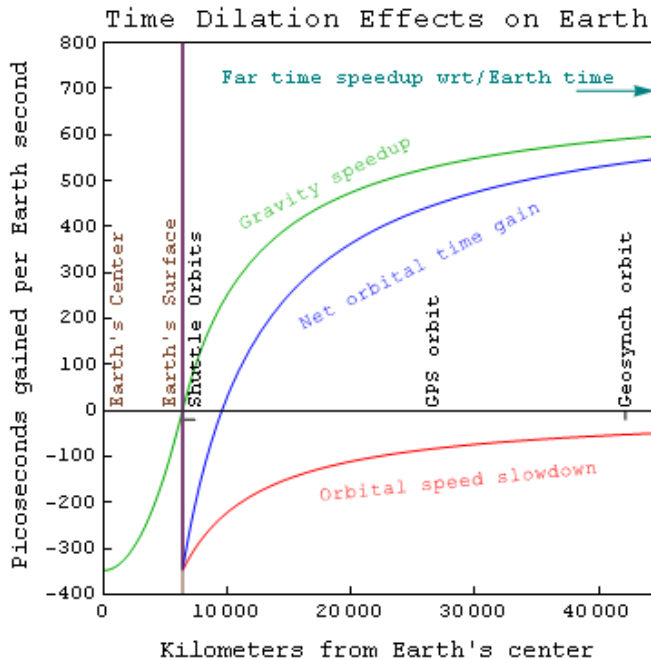


FIG. 2: Time-dilation effects in satellite global-positioning.

On earth's surface the metric equation doesn't change by much since $r_o/r \simeq 1.39117 \times 10^{-9}$, given that **event-horizon radius** $r_o = 2GM/c^2$, where G is the universal gravitation constant and M the earth's mass.

Nonetheless, this modified-metric gives rise to **gravity's geometric-acceleration** of GM/r^2 that at earth's surface becomes $g \simeq 9.8$ meters per second-squared, which must be countered by an upward proper-force of mg (as shown later in this course) to keep a **shell-frame** observer's radius fixed in the neighborhood of a planet. That's because shell-frames (of fixed radius) are not free-float-frames. Around gravitational objects, free-float-frames are sometimes called **rain-frames** instead.

The space-time curvature associated with gravity's geometric-acceleration also distorts space and time. One result of this is the gravitational time-dilation of global-positioning-system (GPS) satellites, as well as of your

head, relative to your boots on the ground.

As with the previous two expressions for differential-aging, this dilation is also linked to an expression involving $\sqrt{1 - 2\text{energy}/mc^2}$, namely:

$$\delta\tau_{\text{radius-r}} \simeq \delta t_{\text{far}} \sqrt{1 - \frac{2GM}{rc^2}}, \quad (10)$$

where potential-energy per unit mass at radius r (also to be shown later in the course) is GM/r . This further means that if clocks at the earth's center and surface began ticking together on the day when earth's formation from the solar-nebula was complete, since then **time-elapsd at earth's center is about a year less than on the surface**. Such differential-aging effects are even more severe with extremely dense objects, like **neutron stars** and the event-horizons of **black holes**.

VI. DISCUSSION

We've not provided all the steps needed to arrive at the standard but "superficially treated" conclusions described here, both to save space and to leave something for curious students to explore. Many of them follow simply from the corresponding metric. More importantly, the results can often be put to use with only the math-tools required for an introductory course!

One caution that may bear repeating is that like time, simultaneity depends on one's choice of reference frame. The metric-equation variables discussed here (including those for speed and acceleration) implicitly assume *simultaneity defined from the vantage point of the map-frame alone*. With that caveat, students inclined to wade through some extra math might be tempted to explore some anyspeed-kinematics as well.

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* pfraundorf@umsl.edu

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⁵ P. Fraundorf, *Path-analysis of metric-first and entropy-first approaches*, arXiv:1106.4698 [physics.gen-ph] (2011), URL <http://arxiv.org/abs/1106.4698>.

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